Jackson 2.14 (a) The series solution is given by

$$\Phi(\rho,\phi) = \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n)$$

The boundary condition is antisymmetric in ϕ, so we have $\alpha_n = 0 \ \forall n$

$$\Rightarrow \quad \Phi(\rho,\phi) = \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi)$$

$$\pi a_n b^n = \int_0^{2\pi} \Phi(\rho=b,\phi)\sin(n\phi)\, d\phi = \cancel{\pi}$$

$$\Rightarrow \quad a_n = \frac{1\cancel{\pi}}{\pi b^n}\int_0^{2\pi} \Phi(\rho=b,\phi)\sin(n\phi)\,d\phi$$

$$= \frac{1\cancel{\pi}}{\pi b^n}\left[\int_0^{\pi/2} V\sin(n\phi)\,d\phi - \int_{\pi/2}^{\pi} V\sin(n\phi)\,d\phi + \int_{\pi}^{3\pi/2} V\sin(n\phi)\,d\phi - \int_{3\pi/2}^{2\pi} V\sin(n\phi)\,d\phi\right]$$

$$= \frac{1\cancel{\pi}}{\pi b^n}\left\{ -\frac{V}{n}\cos(n\phi)\Big|_0^{\pi/2} + \frac{V}{n}\cos(n\phi)\Big|_{\pi/2}^{\pi} - \frac{V}{n}\cos(n\phi)\Big|_{\pi}^{3\pi/2} + \frac{V}{n}\cos(n\phi)\Big|_{3\pi/2}^{2\pi} \right\}$$

$$= \frac{V \cancel{\pi}}{\pi n b^n} \left\{ \begin{array}{l} \cos(n\phi)\Big|_{\pi/2}^{\pi} - \cos(n\phi)\Big|_{0}^{\pi/2} \\ + \cos(n\phi)\Big|_{3\pi/2}^{2\pi} - \cos(n\phi)\Big|_{\pi}^{3\pi/2} \end{array} \right\}$$

$$= \frac{V \cancel{\pi}}{\pi n b^n} \left\{ 2\cos(n\pi) - 2\cos\left(\frac{n\pi}{2}\right) - 2\cos\left(\frac{3n\pi}{2}\right) + 2 \right\}$$

$$= \frac{2 \cancel{\pi} V}{\pi n b^n} \left\{ 1 + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{3n\pi}{2}\right) \right\}$$

$$1 + \cos(n\pi) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{3n\pi}{2}\right) = \begin{cases} 2 & n \text{ divisible by } 4 \\ -2 & n \text{ divisible by 2 only.} \\ 0 & n \text{ odd} \end{cases}$$

$$\Rightarrow \Phi(\rho, \phi) = \frac{4V}{\pi} \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^{4n+2} \frac{\sin[(4n+2)\phi]}{2n+1}$$

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